HYDRODYNAMICS OF HOMOGENEOUS SUSPENSIONS

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The system of dynamic equations describing the average motion of the phases of a monodisperse suspension considered as two interacting interpenetrating continua is obtained. Relations are written which permit determining the average quantities and the correlation functions characterizing the local structure of the flowing suspension.

The general principles underlying the physicomechanical theory of dispersed systems based on statistical analysis of their internal structure (random velocities of the particles and of the fluid phase, and suspension pressure and concentration pulsations) are discussed in [1]. The application of the developed theory to the study of the structure and mechanical behavior of gaseous suspensions, where the calculations can be simplified considerably by neglecting the momentum and viscosity of the gas, has made it possible to achieve good agreement with the available experimental data and to explain several important observed phenomena. Gas suspensions are characterized by a very high level of development of the pulsative ("pseudoturbulent") motions of the phases, and their flows are frequently locally nonhomogeneous, in the sense that in the flows there are formed aggregates containing a large number of particles, bubbles filled only with gas, and so on [2].

Conversely, flows of suspensions of sufficiently fine particles in liquids are usually locally homogeneous and the velocities of the random phase pulsations are relatively small. However, in this case we still can not neglect the inertial forces which arise from acceleration of the liquid or the viscous stresses in the liquid. Therefore, the analysis of such suspensions differs somewhat from that carried out in [1, 2]. Specifically, it is necessary to make a more careful examination of the interaction forces between the two phases, associated with the acceleration of the added liquid masses during relative motion of the particles, and the resulting refinement of the previously obtained dynamic and stochastic equations. This refinement is made in the present study; in so doing the logical scheme of [1, 2] is retained unchanged.

1. DYNAMIC EQUATIONS

In accordance with the method of [1], we represent the particle velocity $\mathbf{w}$, velocity $\mathbf{v}$ of the liquid phase in its specific volume, and the local values of the suspension volumetric concentration ρ and pressure p in the form

$$\langle \mathbf{v}' \rangle = \langle \mathbf{w}' \rangle = \langle \rho' \rangle = \langle p' \rangle = 0$$

$$\mathbf{v} = \langle \mathbf{v} \rangle + \mathbf{v}', \qquad \mathbf{w} = \langle \mathbf{w} \rangle + \mathbf{w}', \qquad \rho = \langle \rho \rangle + \rho', \qquad p = \langle p \rangle + p' \tag{1.1}$$

Here the first terms in the right sides are the ensemble average values of the corresponding quantities (dynamic variables), and the second terms are their random pulsations (pseudoturbulent variables). The ensemble averaging in (1.1) can be pictured as sequential averaging with respect to the conditional distributions $f(p;\, t,\, \mathbf{r} \mid \rho,\, \mathbf{v},\, \mathbf{w})$, $f(\rho;\, t,\, \mathbf{r} \mid \mathbf{v},\, \mathbf{w})$, $f(\mathbf{v};\, t,\, \mathbf{r} \mid \mathbf{w})$ introduced in [1], normed by unity, and then with respect to the particle velocity distribution $f(\mathbf{w};\, t,\, \mathbf{r})$, which is conveniently considered to be normed by the countable particle concentration $n(t,\, \mathbf{r})$. If we perform averaging only with respect to the conditional distributions (denoted below by the superscript degree symbol), we can write

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$$\mathbf{v}^\circ = \langle \mathbf{v} \rangle + \mathbf{v}'', \quad \mathbf{w}^\circ = \langle \mathbf{w} \rangle + \mathbf{w}'', \quad \rho^\circ = \langle \rho \rangle + \rho'', \quad p^\circ = \langle p \rangle + p''$$
$$\langle \mathbf{v}'' \rangle_f = \langle \mathbf{w}'' \rangle_f = \langle \rho'' \rangle_f = \langle p'' \rangle_f = 0, \qquad \mathbf{w}'' \equiv \mathbf{w}'^\circ \tag{1.2}$$

where $\langle \ \rangle_f$ denotes averaging with respect to the distribution $f(\mathbf{w};\, t,\, \mathbf{r})$. For the last terms in the right sides of (1.2), we take

$$\mathbf{v}'' = \mathbf{s}_v \mathbf{w}', \quad \mathbf{w}'' = \mathbf{I}\mathbf{w}', \quad \rho'' = \mathbf{s}_\rho \mathbf{w}', \quad p'' = \mathbf{s}_p \mathbf{w}', \quad \mathbf{I} = \|\delta_{ij}\| \tag{1.3}$$

Here s_v, s_ρ, s_p are some unknown tensors and vectors which can, naturally, depend on the dynamic variables. We note that the assumptions (1.3) are not essential for formulating the dynamic and stochastic equations but are important in finding the function $f(\mathbf{w};\, t,\, \mathbf{r})$. We note that analogous relations can also be written for the other random quantities with double primes.

In order to obtain the dynamic equations defining the average motion of the dispersed phase, we must have the kinetic equation for the unary distribution function $f(\mathbf{w};\, t,\, \mathbf{r})$. As noted in [1], this equation in the conventional sense (not containing higher multiparticle distribution functions) does not exist even for particles suspended in a gas. Therefore, an approximate equation was used in [1], in which the total force $\mathbf{F}_p$ acting on the particle was replaced by its actual average value. Here we shall also use the quantity $\mathbf{F}_p{}^\circ$, obtained by averaging with respect to the conditional distributions, with the objective of obtaining the equation for $f(\mathbf{w};\, t,\, \mathbf{r})$, which plays the role of the kinetic equation "in the mean."

We assume first that direct collisions do not occur in the suspended particle system, and so we introduce the probability $W(\Delta\mathbf{r},\, \Delta\mathbf{w},\, \Delta t \mid \mathbf{r}_0,\, \mathbf{w}_0,\, t_0)$ of particle transfer from the phase volume element $(\mathbf{r}_0,\, \mathbf{r}_0 + d\mathbf{r};\, \mathbf{w}_0,\, \mathbf{w}_0 + d\mathbf{w})$ in which it was located at the moment t_0 into the element $(\mathbf{r},\, \mathbf{r} + d\mathbf{r};\, \mathbf{w},\, \mathbf{w} + d\mathbf{w})$ during the time Δt, where

$$\mathbf{r} = \mathbf{r}_0 + \Delta\mathbf{r}, \qquad \mathbf{w} = \mathbf{w}_0 + \Delta\mathbf{w}, \qquad t = t_0 + \Delta t$$

Clearly, $f(\mathbf{w};\, t,\, \mathbf{r})$ can be written in integral form

$$f(\mathbf{w};\, t,\, \mathbf{r}) = \iint W(\Delta\mathbf{r},\, \Delta\mathbf{w},\, \Delta t \mid \mathbf{r}_0,\, \mathbf{w}_0,\, t_0)\, f(\mathbf{w}_0;\, t_0,\, \mathbf{r}_0)\, d\mathbf{r}_0\, d\mathbf{w}_0$$

Hence we obtain conventionally [3] the equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \mathbf{r}}\left(\frac{f\{\Delta\mathbf{r}\}}{\Delta t}\right) - \frac{\partial}{\partial \mathbf{w}}\left(\frac{f\{\Delta\mathbf{w}\}}{\Delta t}\right) - \frac{1}{2\Delta t}\left[\left(\frac{\partial}{\partial \mathbf{r}} * \frac{\partial}{\partial \mathbf{r}}\right) : \{\Delta\mathbf{r} * \Delta\mathbf{r}\} f\right.$$
$$\left. + 2\left(\frac{\partial}{\partial \mathbf{r}} * \frac{\partial}{\partial \mathbf{w}}\right) : \{\Delta\mathbf{w} * \Delta\mathbf{r}\} f + \left(\frac{\partial}{\partial \mathbf{w}} * \frac{\partial}{\partial \mathbf{w}}\right) : \{\Delta\mathbf{w} * \Delta\mathbf{w}\} f\right] \tag{1.4}$$
$$\mathbf{a} * \mathbf{b} = \|a_i b_j\|, \qquad \mathbf{A} : \mathbf{B} = A_{ij} B_{ji}$$

where the symbol $\{\}$ denotes averaging with respect to the transfer probability, for example,

$$\{\Delta\mathbf{r}\} = \iint \Delta\mathbf{r}\, W(\Delta\mathbf{r},\, \Delta\mathbf{w},\, \Delta t \mid \mathbf{r},\, \mathbf{w},\, t_0)\, d\mathbf{r}_0\, d\mathbf{w}_0$$

Let us examine (1.4) in some particular cases. For a gas of elastic particles the transfer probability is the δ-function, and to within $O((\Delta t^2))$ we have

$$\{\Delta\mathbf{r}\} = \Delta\mathbf{r} = \mathbf{w}\Delta t, \qquad \{\Delta\mathbf{w}\} = \Delta\mathbf{w} = (\mathbf{F}_p/m)\, \Delta t$$

where m is the particle mass, $\mathbf{F}_p$ is independent of $\mathbf{w}$, and all the remaining quantities in the braces in (1.4) are of order $(\Delta t)^2$. For small Δt we obtain from (1.4) the Boltzmann equation without the collisional term.

For a system of Brownian particles we have with the previous accuracy

$$\{\Delta\mathbf{r}\} = \mathbf{w}\Delta t, \qquad \{\Delta\mathbf{w}\} = (\mathbf{F}_p/m)\, \Delta t = -\alpha\mathbf{w}\Delta t, \qquad \{\Delta\mathbf{w} * \Delta\mathbf{w}\} \sim \mathbf{A}\Delta t$$

where A is some ("diffusional") tensor. Using these relations, we obtain the Fokker–Planck equation.

It follows from experiments [4] that in the system in question there are both large-scale pulsations and small-scale "jitter" of the particles and liquid within the limits of their specific volumes, and these pseudoturbulence components can be considered statistically independent. The characteristic time of these components coincides respectively with the "outer" T and "inner" τ pseudoturbulence time scales (see [1, 2] and the references therein), where $T \gg \tau$. Selecting Δt in (1.4) so that $\tau \ll \Delta t \ll T$, we see that in the general case the resulting equation will contain terms which are typical for both the Boltzmann and Fokker–Planck equations. Specifically, there appears in this equation a term describing diffusion in velocity space. It appears that the kinetic equation for suspended particles with account for diffusion in velocity space was first studied in [5] and [6]. The last term does not affect the mass and momentum conservation equations for the dispersed phase, and in this connection it was not introduced in [1]. We shall consider this term here, bearing in mind some further applications.

Converting in the usual way from the set of independent variables t, $\mathbf{r}$, $\mathbf{w}$ to the set t, $\mathbf{r}$, $\mathbf{w}' = \mathbf{w} - \langle \mathbf{w} \rangle$ and introducing the collisional term, we obtain from (1.4) the kinetic equation [7]

$$\frac{Df}{Dt} + \mathbf{w}' \frac{\partial f}{\partial \mathbf{r}} + \frac{\partial}{\partial \mathbf{w}'}\left[\left(\frac{\mathbf{F}_p^\circ}{m} - \frac{D\langle \mathbf{w}\rangle}{Dt}\right) f\right] - \left(\frac{\partial f}{\partial \mathbf{w}'} * \mathbf{w}'\right) : \left(\frac{\partial}{\partial \mathbf{r}} * \langle \mathbf{w}\rangle\right)$$
$$= \frac{1}{m}\left(\frac{\partial}{\partial \mathbf{w}'} * \frac{\partial}{\partial \mathbf{w}'}\right) : (\mathbf{A} f) + \left(\frac{\partial f}{\partial t}\right)_c, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \langle \mathbf{w}\rangle \frac{\partial}{\partial \mathbf{r}} \tag{1.5}$$

The last term in (1.5) describes the change of $f(\mathbf{w};\, t,\, \mathbf{r})$ owing to direct collisions between particles. If the small-scale motions are isotropic (some evidence favoring this hypothesis is presented in [4]), then $\mathbf{A} = A\mathbf{I}$.

If the collisions do not lead to change of the total mass and momentum of the colliding particles, with the aid of standard techniques [7] we obtain the following conservation equations:

$$\frac{\partial \langle \rho\rangle}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\langle \rho\rangle \langle \mathbf{w}\rangle) = \frac{D\langle \rho\rangle}{Dt} + \langle \rho\rangle \frac{\partial \langle \mathbf{w}\rangle}{\partial \mathbf{r}} = 0$$
$$d_2 \langle \rho\rangle \frac{D\langle \mathbf{w}\rangle}{Dt} = -\frac{\partial \mathbf{P}^{(p)}}{\partial \mathbf{r}} + \frac{\langle \rho\rangle}{\sigma_0}\langle \mathbf{F}_p\rangle, \quad \mathbf{P}^{(p)} = d_2 \langle \rho\rangle \langle \mathbf{w}' * \mathbf{w}'\rangle \tag{1.6}$$

Here d_2 is the particle material density, and $\mathbf{P}^{(p)}$ is the internal stress tensor in the dispersed phase. The expression (1.6) for $\mathbf{P}^{(p)}$ was obtained by neglecting the instantaneous nature of momentum transport within the particles themselves. Account for this effect, which plays a significant role in concentrated systems, leads to the following expression for $\mathbf{P}^{(p)}$:

$$\mathbf{P}^{(p)} \approx \varphi d_2 \langle \rho\rangle \langle \mathbf{w}' * \mathbf{w}'\rangle, \quad \varphi = [1 - (\langle \rho\rangle / \rho_*)^{1/3}]^{-1} \tag{1.7}$$

Here ρ_* is the volumetric concentration of the suspension in the dense packing state.

Now let us examine the forces acting on the particle. The force mg acts on the particle from the external mass field, where g is the acceleration of this field. We represent the interaction force of the particle with the surrounding fluid in the form [1, 8]

$$\mathbf{F}_i = -\sigma_0 \frac{\partial p}{\partial \mathbf{r}} + \varkappa m\left[\beta K(\rho)\mathbf{u} + \xi(\rho)\frac{d\mathbf{u}}{dt} + \gamma \int_{-\infty}^{t} \eta(\rho) \frac{d\mathbf{u}}{dt}\Big|_{t=t'} \frac{dt'}{\sqrt{t-t'}}\right] \tag{1.8}$$
$$\sigma_0 = \frac{4}{3}\pi a^3, \quad \beta = \frac{9\nu_0}{2a^2}, \quad \gamma = \frac{9}{2a}\left(\frac{\nu_0}{\pi}\right)^{1/2}, \quad \nu_0 = \frac{\mu_0}{d_1}, \quad \varkappa = \frac{d_1}{d_2}, \quad \mathbf{u} = \mathbf{v} - \mathbf{w}$$

Here differentiation with respect to time is performed along the particle trajectory; a is the particle radius; μ_0 and d_1 are the viscosity and density of the fluid; and K, ξ, η are some functions of ρ, whose specific form is not essential for the purposes of this study. For an isolated Stokesian particle we have

$$K = 1, \; \xi = 1/2, \; \eta = 1$$

In (1.8) no account is taken of the transverse force, which acts even on the Stokesian particle in shear flow [9], or of possible rotation of the particles. Moreover, the viscous interphase interaction force is considered to be linear in the relative velocity $\mathbf{u}$. These approximations are well justified for sufficiently small Reynolds numbers for flow about particles characteristic for the suspensions considered here.

To within terms of second order in the pseudoturbulent variables we obtain from (1.8) the relations

$$\langle \mathbf{F}_i\rangle \approx -\sigma_0 \frac{\partial \langle p\rangle}{\partial \mathbf{r}} + \varkappa m\Big\{\beta\Big[K\langle \mathbf{u}\rangle + \frac{dK}{d\langle \rho\rangle}\langle \rho' \mathbf{u}'\rangle + \frac{1}{2}\frac{d^2K}{d\langle \rho\rangle^2}\langle \mathbf{u}\rangle\langle \rho'^2\rangle\Big]$$
$$+ \xi\frac{D\langle \mathbf{u}\rangle}{Dt} + \frac{d\xi}{d\langle \rho\rangle}\Big\langle \rho'\Big(\frac{d\mathbf{u}'}{dt} + \Big(\mathbf{w}'\frac{\partial}{\partial \mathbf{r}}\Big)\langle \mathbf{u}\rangle\Big)\Big\rangle + \frac{1}{2}\frac{d^2\xi}{d\langle \rho\rangle^2}\frac{D\langle \mathbf{u}\rangle}{Dt}\langle \rho'^2\rangle$$
$$+ \gamma\int_{-\infty}^{t}\Big[\eta\frac{D\langle \mathbf{u}\rangle}{Dt} + \frac{d\eta}{d\langle \rho\rangle}\Big\langle \rho'\Big(\frac{d\mathbf{u}'}{dt} + \Big(\mathbf{w}'\frac{\partial}{\partial \mathbf{r}}\Big)\langle \mathbf{u}\rangle\Big)\Big\rangle + \frac{1}{2}\frac{d^2\eta}{d\langle \rho\rangle^2}\frac{D\langle \mathbf{u}\rangle}{Dt}\langle \rho'^2\rangle\Big]_{t=t'}\frac{dt'}{\sqrt{t-t'}}\Big\} \tag{1.9}$$

$$\mathbf{F}_i' \approx -\sigma_0 \frac{\partial p'}{\partial \mathbf{r}} + \varkappa m \left\{ \beta \left(K\mathbf{u}' + \frac{dK}{d\langle\rho\rangle} \langle \mathbf{u} \rangle \rho' \right) + \xi \left[\frac{d\mathbf{u}'}{dt} + \left(\mathbf{w}' \frac{\partial}{\partial \mathbf{r}} \right) \langle \mathbf{u} \rangle \right] \right.$$
$$\left. + \frac{d\xi}{d\langle\rho\rangle} \frac{D\langle \mathbf{u} \rangle}{Dt} \rho' + \gamma \int_{-\infty}^{t} \left[\eta \left(\frac{d\mathbf{u}'}{dt} + \left(\mathbf{w}' \frac{\partial}{\partial \mathbf{r}} \right) \langle \mathbf{u} \rangle \right) + \frac{d\eta}{d\langle\rho\rangle} \frac{D\langle \mathbf{u} \rangle}{Dt} \rho' \right]_{t=t'} \frac{dt'}{\sqrt{t-t'}} \right\}$$
$$K \equiv K(\langle\rho\rangle), \quad \xi \equiv \xi(\langle\rho\rangle), \quad \eta \equiv \eta(\langle\rho\rangle)$$

In the expression for $\mathbf{F}_i'$ there must in addition appear the sum of terms of the form $a'b' - \langle a'b' \rangle$, where a' and b' are arbitrary pseudoturbulent variables. However, here and in the sequel in the stochastic equations we actually use quantities obtained by averaging over the time interval $\Delta t \gg \tau$ introduced above. The characteristic time for change of the indicated terms with zero average is τ, therefore they drop out in such averaging (see also discussion in [1]).

For the vector $\mathbf{F}_i^\circ$ in (1.5) we have the representation

$$\mathbf{F}_i^\circ = \langle \mathbf{F}_i \rangle + \mathbf{F}_i' \{\mathbf{u}', \rho', p' \to \mathbf{s}_u \mathbf{w}', \ \mathbf{s}_\rho \mathbf{w}', \mathbf{s}_p \mathbf{w}'\}, \qquad \mathbf{s}_u = \mathbf{s}_v - \mathbf{I} \tag{1.10}$$

In the second term in the right side of (1.10) the arguments are replaced in accordance with (1.1)-(1.3).

In deriving (1.6) from (1.5), it is assumed that the momentum of the colliding particles is collision-invariant. In actuality, the velocities of both elastic particles participating in the collision change stepwise as a result of the collision, which leads to stepwise acceleration of the added fluid masses. It is obvious that the force resulting from these jumps is not taken into account in [18]. This force was calculated in [10] for an isolated Stokesian particle; if the particle velocity changes jumpwise by the magnitude Δw at the moment $t = 0$, then

$$F_\delta = \varkappa m \left(\frac{1}{2} \delta(t) + \frac{\gamma}{\sqrt{t}} \right) \Delta w \tag{1.11}$$

If the average time between two successive particle velocity jumps is τ_c, the total particle momentum loss as a result of a single jump is written in the form

$$\Delta M = \int_{-0}^{\tau_c} F_\delta \, dt = \varkappa m \left(\frac{1}{2} + 2\gamma \sqrt{\tau_c} \right) \tag{1.12}$$

This effect is most easily taken into account by "smearing" this momentum loss over time, i.e., by introducing into the expression for $\mathbf{F}_p$ the additional "dissipative" force $\mathbf{F}_d$. The force $\mathbf{F}_d$ has precisely the same sense as, for example, the "diffusive" force introduced in the analysis of diffusion processes by the methods of irreversible process thermodynamics. To determine $\mathbf{F}_d$ we evaluate the total particle energy dissipation in the suspension owing to instantaneous acceleration of the fluid phase during collisions. Examining the collision of a pair of elastic particles in a coordinate system in which one of them is at rest and the other travels with the relative velocity $\mathbf{V}$, forming the angle ψ with the line of centers at the moment of collision, we see that the velocity jump of the moving particle is $2V \sin \psi$ [7]. Accounting for the constrained nature of the motion of replacing the coefficient $1/2$ in (1.12) by ξ, and replacing γ by $\gamma\eta$, we have from (1.12)

$$\Delta M(V, \psi) = \chi m(\xi + 4\gamma\eta \sqrt{\tau_c}) V \sin \psi \tag{1.13}$$

Hence, we obtain also the expression for the work ΔA performed on the fluid as a result of the collision

$$\Delta A(V, \psi) = \varkappa m(\xi + 4\gamma\eta \sqrt{\tau_c}) V^2 \sin^2 \psi \tag{1.14}$$

We obtain the specific particle energy dissipation W_d per unit time owing to the considered phenomenon after averaging (1.14) with respect to V and the collision parameter ψ and multiplying the result by $N/2$ (where $N/n = \tau_c^{-1}$ is the collision frequency), i.e., we have from (1.14)

$$W_d = \frac{N}{2} \left\langle 2 \int_0^1 \Delta A \cos \psi d(\cos \psi) \right\rangle = \frac{c}{4} \varkappa m N \left(\xi + \frac{4\gamma\eta \sqrt{n}}{\sqrt{N}} \right) \langle w'^2 \rangle \tag{1.15}$$

Here we have taken $\langle V^2\rangle = c\langle w'^2\rangle$. Representing W_d in the form of a sum over the particles in unit volume, we have from (1.15)

$$W_d = -\sum_{j=1}^{n} \mathbf{F}_d^{(j)}\mathbf{w}'^{(j)}, \quad \mathbf{F}_d = -\varkappa m\zeta\mathbf{w}', \quad \zeta = \frac{c}{4}\frac{N}{n}\left(\xi + 4\gamma\eta\sqrt{\frac{n}{N}}\right) \tag{1.16}$$

The parameters c and N introduced above can be approximately evaluated by using the corresponding expressions which hold for a dense gas with isotropic velocity distribution [7]:

$$c = 2, \qquad N = 16a^2n^2\left(\frac{\pi}{3}\langle w'^2\rangle\right)^{1/2}\chi, \qquad n = \frac{\langle\rho\rangle}{\sigma_0} \tag{1.17}$$

Here the coefficient χ shows how many times the binary collision frequency increases in the system of particles of volume $\sigma_0 \neq 0$ in comparison with the system of point particles. Using the Enskog results [7, 11], for small and large $\langle\rho\rangle$ we obtain the following expressions for χ for the dense gas of elastic spheres:

$$\chi \approx \frac{1 - {}^{11}/_2\langle\rho\rangle}{1 - 8\langle\rho\rangle}, \quad \langle\rho\rangle \ll \rho_*, \qquad \chi \approx \frac{1}{4\langle\rho\rangle^{2/3}(\rho_*^{1/3} - \langle\rho\rangle^{1/3})}, \quad \langle\rho\rangle \sim \rho_* \tag{1.18}$$

From (1.16) and (1.17) we obtain also the estimate for ζ:

$$\zeta \approx \left(\frac{3}{\pi}\langle w'^2\rangle\right)^{1/4}\left(\frac{\langle\rho\rangle\chi}{a}\right)^{1/2}\left[\xi\left(\frac{3}{\pi}\langle w'^2\rangle\right)^{1/4}\left(\frac{\langle\rho\rangle\chi}{a}\right)^{1/2} + 2\gamma\eta\right] \tag{1.19}$$

We note that the expressions obtained here can be considered valid only for locally homogeneous dispersed systems, when it is admissible to consider the particles independently of one another. However, local nonhomogeneity is characteristic basically for gas suspension flows, where within the framework of the approximation $\varkappa = 0$ (but $\beta\varkappa \neq 0$) used in [1, 2] the force $\mathbf{F}_d$ from (1.16) can in general be neglected.

In deriving the dynamic equations for the fluid phase, we write the equations of motion of the fluid through the lattice of pulsating particles in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\rho - (1-\rho)\frac{\partial \mathbf{v}}{\partial \mathbf{r}} = 0, \quad d_1(1-\rho)\left(\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{v} = -\frac{\partial p}{dr}$$
$$+\frac{\partial(\mu\mathbf{e})}{\partial\mathbf{r}} + d_1(1-\langle\rho\rangle)\,\mathbf{g} - \frac{\langle\rho\rangle}{\sigma_0}\mathbf{F}_i, \quad \mathbf{e} = \left\|\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_l}{\partial x_l}\right\|, \quad \mu = \mu_0 S(\rho) \tag{1.20}$$

Here μ is the effective viscosity of the fluid phase flowing through the particle lattice. [The function $S(\rho)$ was calculated in [12] on the basis of the cellular model of constrained particle flow. V. M. Safrai, using altered versions of this model, obtained recently somewhat different expressions for $S(\rho)$.] The viscous term is introduced into (1.20) so that the "compressibility" of the fluid phase owing to changes of suspension concentration does not lead to any energy dissipation. The term with the viscous stresses in (1.20) is not multiplied by $1 - \rho$, since by the sense of its definition the viscosity μ describes the stresses referred to the volume of the mixture, and not to the volume of the fluid alone [10, 12].

Averaging (1.20), we obtain, as in [1], the dynamic equations for the fluid phase

$$\left(\frac{\partial}{\partial t} + \langle\mathbf{v}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\langle\rho\rangle - (1-\langle\rho\rangle)\frac{\partial\langle\mathbf{v}\rangle}{\partial\mathbf{r}} - \frac{\partial\mathbf{q}}{\partial\mathbf{r}} = 0, \quad \mathbf{q} = -\langle\rho'\mathbf{v}'\rangle$$
$$d_1\left[\frac{\partial}{\partial t}((1-\langle\rho\rangle)\langle\mathbf{v}\rangle) + \frac{\partial}{\partial\mathbf{r}}((1-\langle\rho\rangle)\langle\mathbf{v}\rangle * \langle\mathbf{v}\rangle) + \frac{\partial\mathbf{q}}{\partial t}\right]$$
$$= -\frac{\partial\mathbf{P}^{(f)}}{\partial\mathbf{r}} - \frac{\partial\langle p\rangle}{\partial\mathbf{r}} + \mu_0\frac{\partial}{\partial\mathbf{r}}\left(S\langle\mathbf{e}\rangle + \frac{dS}{d\langle\rho\rangle}\langle\rho'\mathbf{e}'\rangle + \frac{1}{2}\frac{d^2S}{d\langle\rho\rangle^2}\langle\mathbf{e}\rangle\langle\rho'^2\rangle\right)$$
$$+ d_1(1-\langle\rho\rangle)\,\mathbf{g} - \langle\rho\rangle\frac{\langle\mathbf{F}_i\rangle}{\sigma_0}, \quad S \equiv S(\langle\rho\rangle) \tag{1.21}$$
$$\mathbf{P}^{(f)} = d_1[(1-\langle\rho\rangle)\langle\mathbf{v}' * \mathbf{v}'\rangle + \mathbf{q} * \langle\mathbf{v}\rangle + \langle\mathbf{v}\rangle * \mathbf{q}]$$

A discussion of the sense of the various terms in (1.21) is presented in [1].

The different pseudoturbulent averages appearing in (1.6) and (1.21) must, naturally, be expressed in terms of the dynamic variables. Usually solutions of equations of the type (1.5) are used for this purpose; however, in the present case this cannot be done since, first, (1.5) does not contain information on

$\mathbf{v}'$, ρ', p' and, second, it depends on the unknown tensor **A**. Therefore, here, as in [1, 2], we use stationary random process correlation theory to determine the pseudoturbulent characteristics.

2. STOCHASTIC EQUATIONS

Following the method of [1], the stochastic equations for pseudoturbulent pulsations are obtained from the equation of motion of some particle and the equations (1.20), after subtracting from them the corresponding averaged equations and averaging the result over the time interval $\Delta t > \tau$. In so doing, in [1] the time derivatives along the particle trajectories were transformed using the formula

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{w}\frac{\partial}{\partial \mathbf{r}}$$

although, for example, $(\boldsymbol{\omega}\partial/\partial\mathbf{r})\boldsymbol{\omega}$ does not have a clear physical meaning. Here we identify in the time derivatives of different type precisely the derivative d/dt, for example,

$$\frac{\partial}{\partial t} + \mathbf{v}\frac{\partial}{\partial \mathbf{r}} = \frac{d}{dt} + \mathbf{u}\frac{\partial}{\partial \mathbf{r}}$$

Then we obtain from (1.20) the stochastic equations

$$\left(\frac{d}{dt} + \langle\mathbf{u}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\rho' + \frac{\partial\langle\mathbf{v}\rangle}{\partial\mathbf{r}}\rho' - (1-\langle\rho\rangle)\frac{\partial\mathbf{v}'}{\partial\mathbf{r}} + \frac{\partial\langle\rho\rangle}{\partial\mathbf{r}}\mathbf{u}' = 0$$

$$d_1(1-\langle\rho\rangle)\left(\frac{d}{dt} + \langle\mathbf{u}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{v}' + d_1(1-\langle\rho\rangle)\left(\mathbf{u}'\frac{\partial}{\partial\mathbf{r}}\right)\langle\mathbf{v}\rangle - d_1\rho'\left(\frac{\partial}{\partial t} + \langle\mathbf{v}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\langle\mathbf{v}\rangle \tag{2.1}$$

$$= -\frac{\partial p'}{\partial\mathbf{r}} + \mu_0\frac{\partial}{\partial\mathbf{r}}\left(S\mathbf{e}' + \frac{dS}{d\langle\rho\rangle}\langle\mathbf{e}\rangle\rho'\right) - \frac{\langle\rho\rangle}{\sigma_0}\mathbf{F}_i'$$

Further, the particle equation of motion has the form

$$m\frac{d\mathbf{w}}{dt} = \mathbf{F}_i + \mathbf{F}_d + \mathbf{F}_c + mg = \mathbf{F}_p \tag{2.2}$$

where $\mathbf{F}_C$ is the random force acting during collision of the identified particle with the neighboring particles and disappearing upon averaging over Δt. After a simple transformation we obtain from (2.2) still another stochastic equation

$$m\frac{d\mathbf{w}'}{dt} = \mathbf{F}_i' + \mathbf{F}_d', \qquad \mathbf{F}_d' \equiv \mathbf{F}_d \tag{2.3}$$

Thus, all the random functions are examined in the coordinate system traveling together with the particle. In the zero approximation with respect to the derivatives of the dynamic quantities, the approach in this study is identical to that of [1], where the analysis was made in a coordinate system traveling together with the average dispersed phase flux.

We represent all the random functions in the form of Fourier–Stieltjes integrals, for example,

$$\rho' = \int e^{i(\omega t+\mathbf{kr})}dZ_\rho, \quad \mathbf{v}' = \int e^{i(\omega t+\mathbf{kr})}d\mathbf{Z}_v \tag{2.4}$$

Substituting relations of the type (2.4) into (2.1) and (2.3), we obtain the equations for the spectral measures in terms of Fourier–Stieltjes integrals

$$\left[i(\omega + \langle\mathbf{u}\rangle\mathbf{k}) + \frac{\partial\langle\mathbf{v}\rangle}{\partial\mathbf{r}}\right]dZ_\rho - i(1-\langle\rho\rangle)\mathbf{k}\,d\mathbf{Z}_v + \frac{\partial\langle\rho\rangle}{\partial\mathbf{r}}(d\mathbf{Z}_v - d\mathbf{Z}_w) = 0$$

$$d_1(1-\langle\rho\rangle)\left[i(\omega + \langle\mathbf{u}\rangle\mathbf{k})\,d\mathbf{Z}_v + \left((d\mathbf{Z}_v - d\mathbf{Z}_w)\frac{\partial}{\partial\mathbf{r}}\right)\langle\mathbf{v}\rangle\right] - d_1\left(\frac{\partial\langle\mathbf{v}\rangle}{\partial t} + \left(\langle\mathbf{v}\rangle\frac{\partial}{\partial\mathbf{r}}\right)\langle\mathbf{v}\rangle\right)dZ_\rho \tag{2.5}$$

$$= -i\mathbf{k}\,dZ_p + \mu_0\left\{-S\left[k^2 d\mathbf{Z}_v + \frac{1}{3}\mathbf{k}\,(\mathbf{k}d\mathbf{Z}_v)\right] + \frac{dS}{d\langle\rho\rangle}\left[\left(i\langle\mathbf{e}\rangle\mathbf{k} + \frac{\partial\langle\mathbf{e}\rangle}{\partial\mathbf{r}}\right)dZ_\rho\right.\right.$$

$$\left.\left. + i\left(\frac{\partial\langle\rho\rangle}{\partial\mathbf{r}}\left(\mathbf{k}*d\mathbf{Z}_v + d\mathbf{Z}_v*\mathbf{k} - \frac{2}{3}(\mathbf{k}\,d\mathbf{Z}_v)\,\mathbf{I}\right)\right)\right] + \frac{d^2S}{d\langle\rho\rangle^2}\frac{\partial\rho}{\partial\mathbf{r}}\langle\mathbf{e}\rangle\,dZ_\rho\right\}$$

$$-\frac{\langle\rho\rangle}{\sigma_0}d\mathbf{Z}_F^{(i)} - id_2\omega\,d\mathbf{Z}_w = \sigma_0^{-1}(d\mathbf{Z}_F^{(i)} + d\mathbf{Z}_F^{(d)})$$

We obtain the relations for $dZ_F^{(i)}$ and $dZ_F^{(d)}$ from (1.9) and (1.16)

$$
\begin{aligned}
dZ_F^{(i)} = &- i\sigma_0 \mathbf{k} d\mathbf{Z}_p + d_1\sigma_0 \Big\{\beta\Big[K(d\mathbf{Z}_v - d\mathbf{Z}_w) + \frac{dK}{d\langle\rho\rangle}\langle\mathbf{u}\rangle dZ_\rho\Big] \\
&+ \xi\Big[i\omega(d\mathbf{Z}_v - d\mathbf{Z}_w) + \Big(d\mathbf{Z}_w\frac{\partial}{\partial\mathbf{r}}\Big)\langle\mathbf{u}\rangle\Big] + \frac{d\xi}{d\langle\rho\rangle}\frac{D\langle\mathbf{u}\rangle}{Dt}dZ_\rho \\
&+ \gamma\Big[r_1(\omega)(d\mathbf{Z}_v - d\mathbf{Z}_w) + \Big(d\mathbf{Z}_w\frac{\partial}{\partial\mathbf{r}}\Big)\mathbf{r}_2(\omega) + \mathbf{r}_3(\omega)dZ_\rho\Big]\Big\} \\
dZ_F^{(d)} = &- d_1\sigma_0\zeta\, d\mathbf{Z}_w
\end{aligned}
\tag{2.6}
$$

The quantities $\mathbf{r}_1(\omega)$, $\mathbf{r}_2(\omega)$, $\mathbf{r}_3(\omega)$ are represented in the form

$$
\begin{aligned}
r_1(\omega) = i\omega\int_0^\infty \eta|_{t-\tau}\, e^{-i\omega\tau}\frac{d\tau}{\sqrt{\tau}}, \quad \mathbf{r}_2(\omega) = \int_0^\infty \eta\langle\mathbf{u}\rangle|_{t-\tau} e^{-i\omega\tau}\frac{d\tau}{\sqrt{\tau}} \\
\mathbf{r}_3(\omega) = \int_0^\infty \frac{d\eta}{d\langle\rho\rangle}\frac{D\langle\mathbf{u}\rangle}{Dt}\Big|_{t-\tau} e^{-i\omega\tau}\frac{d\tau}{\sqrt{\tau}}
\end{aligned}
\tag{2.7}
$$

Equations (2.5) with account for (2.6), (2.7) permit expressing the seven spectral measures dZ_{vi}, dZ_{wi}, dZ_p (i = 1, 2, 3) in terms of the spectral measure dZ_ρ, and thereby determine the statistical characteristics of all the random processes in terms of the process spectral density ρ'. The expression for the latter is obtained with the aid of the generalized diffusion equation, derived in [1]. Retaining in this equation those terms depending on the derivatives of the dynamic quantities, which were neglected in [1], we obtain following [1] the relation

$$
\begin{aligned}
\Psi_{\rho,\rho}(\omega,\mathbf{k}) = \frac{\langle dZ_\rho^* dZ_\rho\rangle}{d\omega\, d\mathbf{k}} = \frac{\Phi_{\rho,\rho}(\mathbf{k})}{M(\omega,\mathbf{k})}\Big(\int\frac{d\omega}{M(\omega,\mathbf{k})}\Big)^{-1} \\
M(\omega,\mathbf{k}) = \Big(\omega - \mathbf{k}\frac{\partial\mathbf{D}}{\partial\mathbf{r}}\Big)^2 + \Big(\mathbf{Dkk} - \frac{\operatorname{tr}\mathbf{D}}{\langle w'^2\rangle}\omega^2\Big), \quad \operatorname{tr}\mathbf{D} = D_{ii}
\end{aligned}
\tag{2.8}
$$

Here $\mathbf{D}$ is the particle diffusion coefficient tensor, and for the particle spectral density $\Phi_{\rho,\rho}(\mathbf{k})$ in the system of statistically independent particles, we have the approximate expressions [1]

$$
\begin{aligned}
\Phi_{\rho,\rho}(\mathbf{k}) &\approx \frac{3\sigma_0}{8\pi^3}\langle\rho\rangle\Big(1 - \frac{\langle\rho\rangle}{\rho_*}\Big)\frac{\sin kb_0 - kb_0\cos kb_0}{(kb_0)^3} \\
\Phi_{\rho,\rho}(\mathbf{k}) &\approx \frac{3}{4\pi}\frac{\langle\rho\rangle^2}{k_0^3}\Big(1 - \frac{\langle\rho\rangle}{\rho_*}\Big)Y(k_0 - k), \quad Y(x) = \begin{cases}1, & x > 0\\ 0, & x < 0\end{cases} \\
b_0 &= \frac{a}{\langle\rho\rangle^{1/3}}\Big(1 - \frac{\langle\rho\rangle}{\rho_*}\Big)^{1/3}, \quad k_0 = \Big(\frac{3\pi}{2}\Big)^{1/3}\frac{1}{b_0}
\end{aligned}
\tag{2.9}
$$

Relations (2.8), (2.9), together with the mentioned representations of all the random measures in terms of dZ_ρ, make it possible to find the pseudoturbulent averages which are of interest in the theory. The averages determined in this way will, of course, depend not only on the dynamic variables and the physical characteristics of both phases, but also on all the pseudoturbulent averages encountered in the stochastic equations and in (2.8). To express these averages in terms of the dynamic variables we use the obvious equations

$$
\langle a'b'\rangle = \iint\Psi_{a,b}(\omega,\mathbf{k})\,d\omega\, d\mathbf{k} = \int\langle dZ_a^*\, dZ_b\rangle
\tag{2.10}
$$

where a' and b' are any pseudoturbulent variables. Moreover, the a priori unknown components of the diffusion tensor $\mathbf{D}$ appear in (2.8). Representing them in the usual way in terms of the integrals of the corresponding Lagrangian correlation functions for the particle velocity, we obtain as a result the equations

$$
E_{ij} = \frac{1}{2}\int_0^\infty d\tau\iint e^{i\omega\tau}(\Psi_{wi,wj}(\omega,\mathbf{k}) + \Psi_{wj,wi}(\omega,\mathbf{k}))\,d\omega\, d\mathbf{k}
\tag{2.11}
$$

An example of the calculation of $\langle w'^2\rangle$ and D_{ij} from (2.10) and (2.11) and further investigation of the dynamic equations can be found in [2].

Further, we have the system of linear algebraic equations for the quantities s_r, s_ρ, s_p appearing in the kinetic equation (1.5)

$$
\langle\rho' w_i'\rangle = s_{\rho j}\langle w_j' w_i'\rangle, \quad \langle p' w_i'\rangle = s_{pj}\langle w_j' w_i'\rangle \quad \langle v_i' w_j'\rangle = s_{vil}\langle w_l' w_j'\rangle, \qquad \mathbf{s}_u = \mathbf{s}_v - \mathbf{I}
\tag{2.12}
$$

Assuming that the tensor **A** in (1.5) is independent of **w**', we can in principle find the solution (1.5), which depends on A as a parameter. Equating the known expressions for $\langle w_i' w_j' \rangle$ to the corresponding expressions obtained directly from the distribution function, we then obtain the system of equations for finding the components of the tensor **A**. Thus, the distribution function can also be expressed in terms of only the dynamic variables and the physical parameters of the phases. Specifically, it can be used to refine the parameters c and N in (1.16).

We note that the entire theory proposed is meaningful, naturally, only for $T \ll T_0$, $L \ll L_0$, where T and L are the pseudoturbulence time and space scales, and T_0, L_0 are the corresponding average flow scales. A similar situation holds in the kinetic theory of gases. It is the satisfaction of these inequalities which makes it possible, in particular, to assume that the spectral measures introduced above depend on t and **r** (implicitly, through the dynamic variables) so weakly that the use of the mathematical apparatus of stationary random processes is admissible.

The concrete calculations using the scheme proposed in Section 2 are in most cases very tedious and time consuming. Therefore, it is advisable in the future to examine successive approximations in the small ratios T/T_0, L/L_0. Such approximations of zero, first, and second order respectively have the same meaning as do the hydrodynamic approximations of Euler, Navier–Stokes, and Burnett.

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